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# Numerical Study of Laminar Boundary Layer Using Navier-Stokes Equation and Finite Volume Method

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## Abstract

*The boundary layer is known as the distance from the surface to a particular point perpendicular to the direction of flow where the flow velocity has retained 99% of the free stream velocity providing 'no-slip' condition at the surface. It can be laminar or turbulent and there is a zone of 'transition' from laminar to turbulent depending on Reynolds number. In this paper the properties of laminar boundary layer such as development of velocity profile along the flow direction, boundary layer thickness, displacement thickness, momentum thickness, shape factor, wall shear stress, friction coefficient, drag coefficient etc. for 5 m/s flow over a smooth flat plate of 1 meter are studied by exact solution of Blasius's equation and compared with computer-aided CFD solution using Navier-Stokes equations and Finite Volume method. It is found that different boundary layer thicknesses are direct function of local Reynolds number and these thicknesses increase as the flow travels further downstream. Moreover the shear stress and subsequently, the local skin friction reduce as the flow travels in downstream direction. The shape factor is found to be 2.590361 which is close the value obtained from Blasius's solution.*

***Keywords:*** *Blasisus's equation, boundary layer thickness, displacement thickness, momentum thickness, shape factor, friction coefficient.*

## 1. Introduction

At the interface between a fluid and a surface in relative motion, a condition known as 'no slip' dictates an equivalence between fluid and surface velocities. Away from the surface, the fluid velocity rapidly increases; the zone in which this occurs is known as the boundary layer. The boundary layer is the thin region of flow adjacent to a surface, where the flow is retarded by the influence of friction between a solid surface and fluid. Although the boundary layer occupies geometrically only a small portion of flow field, its influence on different aerodynamic and heat transfer phenomena to the body is immense as Prandtl described as 'marked results' [1].Smooth thin flat plate has long been considered to be simplest form to describe boundary layer as there is no pressure gradient involved and it was probably the first example illustrating the application of Prandtl's boundary layer theory.

Shear stress acts as a pivotal parameter for the existence of boundary layer. The shear stress on the smooth surface is a direct function of the velocity gradient at the surface of the plate. This shear stress acting at the plate surface sets up a shear force which opposes the fluid motion and fluid close to the wall is decelerated. If the flow travels further along the surface, at zero pressure gradients, the shear force is effectively increased due to the increased plate surface wetted area. More and more of fluid retarded and the thickness of the fluid layer increases. Reynolds number (Re) can be considered as the measure-stickfor behavior of the boundary layer. If the Re; calculated locally is low, the fluid flow close to the wall may be categorized as laminar. For smooth, polished plates the transition from laminar to turbulent may be delayed until Re 500000 i.e. below this Re the flow can be considered as laminar. However, for rough plates or for turbulent approach flows, transition may occur at much lower values.

There are number of intriguing properties of boundary layer which are decisive for analyzing different flow phenomena like drag or shear stress. These properties can be expressed through mathematical expressions which are direct function of local Re and distance of the point under consideration on the plate from the leading edge. Boundary layer thickness $\delta$ is the distance from the surface of the plate in perpendicular direction up to a point where the velocity of the flow is 99% of the free stream velocity. Displacement thickness $\delta^*$ can be considered as missing mass flow which is the difference between actual mass flow and hypothetical mass flow through the boundary layer if the boundary layers were not present. Another boundary layer property of importance is the momentum thickness θ, which is an index that is proportional to the decrement in momentum flow due to the presence of the boundary layer. It is the height of a hypothetical streamtube which is carrying the missing momentum flow at free stream conditions. Shape factor H of velocity profile is the ratio of the displacement thickness to the momentum thickness which increases in an adverse pressure gradient. For laminar flow with zero pressure gradient (such as a flat plate), it is 2.59 and it reaches to 3.5 at separation. Local friction coefficient $C_{f,x}$is the dimensionless number defined as the ratio of wall shear stress to dynamic pressure.

Blasius [2] was the first one to illustrate Prandtl's boundary layer theory through the application of flow over a flat plate. He provided the legendary equation known as 'Blasius's equation'. Bairstow [3], Goldstein [4] solved it through analytical procedure while Töpfer [5] solved it using Runge-Kutta numerical method. Howarth [6] solved the equation with greater accuracy using numerical procedure. Steinheuer [7] published a systematic review of the solutions to Blasius's equation.Filobello-Nino et. al. [8] provided with an approximate solution of Blasius's equation by using HPM (Homotopy Perturbation Method) and described the behavior of a two-dimensional viscous laminar flow over flat plate. Aminikhah [9] persuaded analytical approximation to the solution of non-linear Blasius's viscous flow equation by LTNHPM (Laplace Transform and New Homotopy Perturbation Method). In this paper the laminar boundary layer properties are illustrated using exact solution of Blasius's equation and these properties are analyzed using flow over one side of a smooth flat plate with no pressure gradient by solving the Navier-Stokes equation set using the Finite Volume Method.

## 2. Mathematical Model

Incompressible, two dimensional flows over a thin flat plate at 0° angle of incidence is simplest example used in the first place to describe Prandtl's boundary layer theory. For such flow the density and viscosity are constant and the pressure gradient is zero as inviscid flow over the smooth flat plate at 0° angle of attack yields constant pressure over the surface. Thus the Navier-Stokesequations reduce to:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$\frac{\partial p}{\partial y}=0 \tag{3}$$

Here $\nu$ is the kinematic viscosity defined as $\nu = \mu/\rho$. The exact solution is described by Blasius [2]; a student of Prandtl in his doctor's thesis at Goettingon. The independent variable (x, y) are then transformed into *( ξ, η)* as $\xi = x$ and $\eta = y\sqrt{\frac{V_\infty}{x\,\nu}}$and the stream function is considered to be $\psi = f(\eta)\times\sqrt{x\,\nu V_\infty}$ [10], [11], [12] where $f$ is strictly a function of $\eta$. Blasius concluded with a legendary equation known as Blasius's Equation' as form of

$$2f'''+ff''=0 \tag{4}$$

Where the function $f(\eta)$ has the property that is $f'$ is described as $\frac{u}{V_\infty}$ . This is a third order non-linear differential equation which requires three boundary conditions to solve which are:at $\eta = 0 : f = 0, f' = 0$ and $\eta = \infty : f' = 1$. The equation was solved by Blasius using a series of approach. The properties of boundary layer are defined as in Table 1.

**Table 1.**Properties of laminar boundary layer over flat plate

| Boundary layer property | Mathematical expression |
|---|---|
| Boundary layer thickness (δ) | $5x/\sqrt{Re_x}$ |
| Displacement thickness (δ*) | $1.72x/\sqrt{Re_x}$ |
| Momentum thickness (θ) | $0.664x/\sqrt{Re_x}$ |
| Shape factor (H) | $\delta^*/\theta$ |
| Friction coefficient ($C_{f,x}$) | $0.664/\sqrt{Re_x}$ |
| Drag coefficient ($C_D$) | $1.328/\sqrt{Re_L}$ |

Here $Re_x$ refers to be local Reynolds number and $Re_L$ is the overall Reynolds number. To calculate the $Re_x$, the distance is measured from the leading edge of the flat plate. In case of $Re_L$, the distance is the total length of the plate. The length of the plate is kept 1 m because the development of laminar boundary layer is expected to be completed within this length. Longer plate length would increase the size of computational domain thus more computational time would be required.

## 3. Numerical Procedure

Equation set consisting equation no (1), (2) and (3) are solved using 'Finite Volume' method. The cell-centered finite volume (FV) method is used to obtain conservative approximations of the governing equations on the locally refined rectangular mesh. The governing equations are integrated over a control volume which is a grid cell, and then approximated with the cell-centered values of the basic variables. The integral conservation laws may be represented in the form of the cell volume and surface integral equation:

$$\frac{\partial}{\partial t}\int \mathbf{U}\,\mathrm{dv} + \oint F.ds = \int Q dv \text{ , which is replaced by } \frac{\partial}{\partial t}(U\mathrm{v}) + \sum_{cell\ faces} F.s = Qv$$

The second-order upwind approximations of fluxes Fare based on the implicitly treated modified Leonard's QUICK approximations [13] and the Total Variation Diminishing (TVD) method [14].The rectangular computational domain is constructed, so it encloses the solid body and has the boundary planes orthogonal to the specified axes of the Cartesian coordinate system. Then, the computational mesh is constructed in the following several stages. First of all, a basic mesh is constructed. For that, the computational domain is divided into slices by the basic mesh planes, which are evidently orthogonal to the axes of the Cartesian coordinate system. The basic mesh is determined solely by the computational domain and does not depend on the solid/fluid interfaces. Then, the basic mesh cells intersecting with the solid/fluid interface are split uniformly into smaller cells in order to capture the solid/fluid interface with mesh cells of the specified size i.e. with respect to the basic mesh cells. The following procedure is employed: each of the basic mesh cells intersecting with the solid/fluid interface is split uniformly into 8 child cells; each of the child cells intersecting with the interface is in turn split into 8 cells of next level, and so on, until the specified cell size is attained. At the next stage of meshing, the mesh obtained at the solid/fluid interface with the previous procedure is refined (i.e. the cells are split further or probably merged) in accordance with the solid/fluid interface curvature. The criterion to be satisfied is established as follows: the maximum angle between the normals to the surface inside one cell should not exceed certain threshold; otherwise the cell is split into 8 cells. As a result of all these meshing procedures, a locally refined rectangular computational mesh is obtained and used then for solving the governing equations on it.

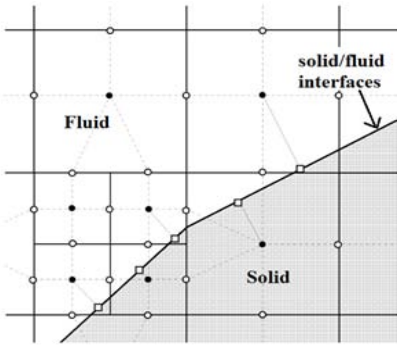


**Fig. 1.**Computational mesh type

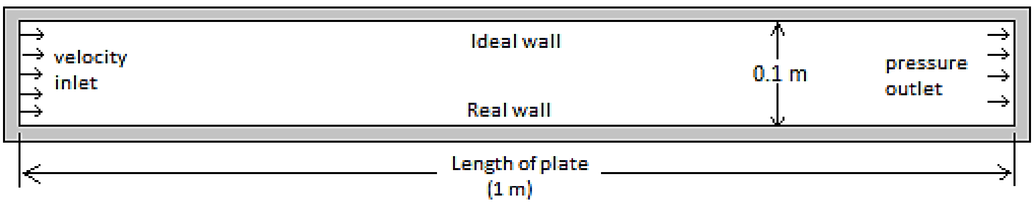


**Fig. 2.**Computational design set-up

The ‘inlet velocity’of air is 5 m/s at 293.2 K and 101325 Pa conditions. ‘Pressure opening’ at the outlet is at 101325 Pa. Upper wall of the design is ‘Ideal Wall’ while the lower wall of 1 m length is ‘Real Wall’ i.e. ‘no-slip’ condition is applied. Needless to say that the lower wall is considered as the smooth flat plate.

## 4. Results and Discussion

For air flow over a smooth flat plate of 1 meter length without any pressure gradient and heat transfer, the local Re never crossed the critical Re (500000) that could cause transition of laminar flow to turbulent flow. So the boundary layer generated at the vicinity of the lower wall of the computational design can be considered as laminar boundary layer. At the area closer to the leading edge of the plate, the boundary layer thickens rapidly (Fig. 3 and Fig. 4 (a), (b), (c)). As the flow travels further downstream the rate of thickening of the boundary layer decreases and at some points around 70-90% of the plate length, the thickening effect becomes more obscure (Fig. 4(e)). With increasing distance from leading edge, the point at which the local velocity of the flow becomes almost equal to the free stream velocity; travels more to the perpendicular direction of the plate i.e. y direction. Both from Fig. 3 and Fig. 5 it is evident that the thickness of the boundary layer δ increases as the flow travels more downstream because more and more fluid particles pile up due to increase of wall shear stress at that direction.

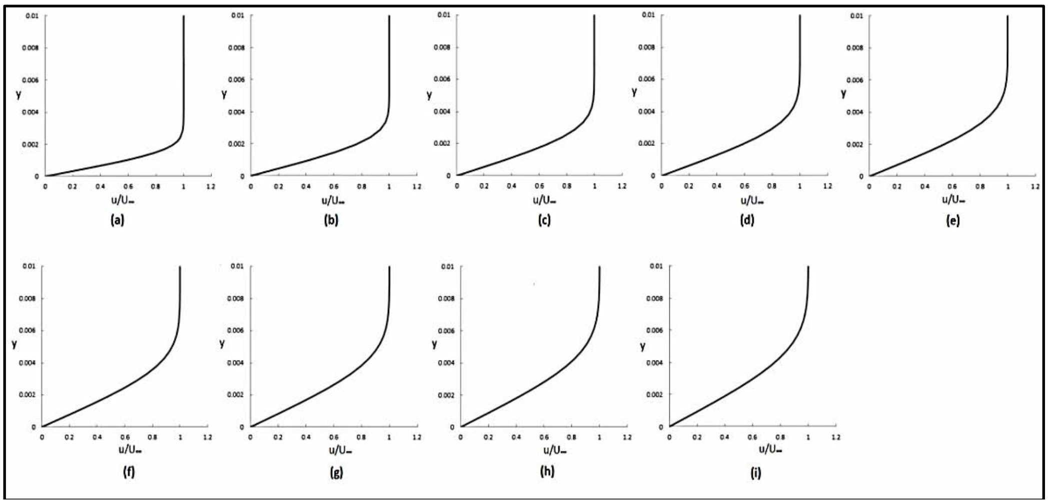


**Fig. 3.**Velocity profile at (a) x=0.1m, (b) x=0.2m, (c) x=0.3m, (d) x=0.4m, (e) x=0.5m, (f) x=0.6m, (g) x=0.7m, (h) x=0.8m,(i) x=0.9m. The perpendicular distance from the plate (y- axis) is in ‘m’.

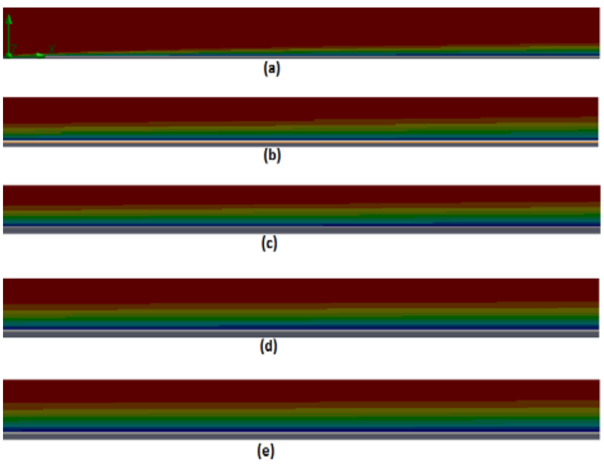


**Fig. 4.** Development of boundary layer at (a) x=0.0 to 0.1, (b) x=0.1 to 0.2, (c) x=0.2 to 0.3, (d) x=0.4 to 0.5 and (e) x= 0.7 to 0.8.

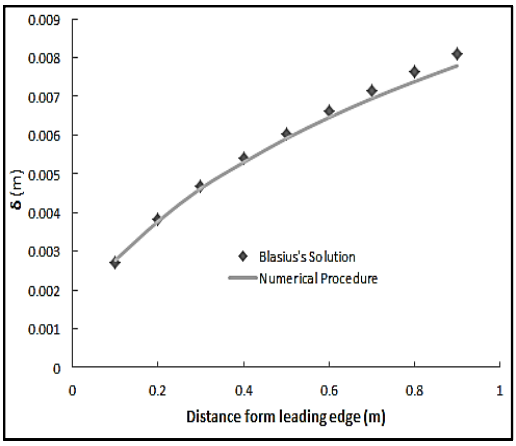


**Fig. 5.** Boundary layer thickness (δ) at different position of the plate.

Boundary layer thickness is so far referred to only in physical terms. It is however, possible to define boundary layer thickness in terms of the effect on the flow. Displacement thickness is defined as the 'distance' the surface would have to move in the y direction to reduce the flow passing by a volume equivalent to the real effect of the boundary layer. Displacement thickness $\delta^*$ for the boundary layer increases with increasing distance from the leading edge of the plate (Fig.6).With increasing distance from the leading edge of the plate, $\delta^*$ increases due to the same reason as $\delta$ increases. That means the plate would have to move further in y direction in case there is no boundary layer to compensate the flow reduction due to boundary layer. Similar outcome is found for momentum thickness, $\theta$ as decrement in momentum flow due to the boundary layer increases as the flow travels further downstream (Fig.7).

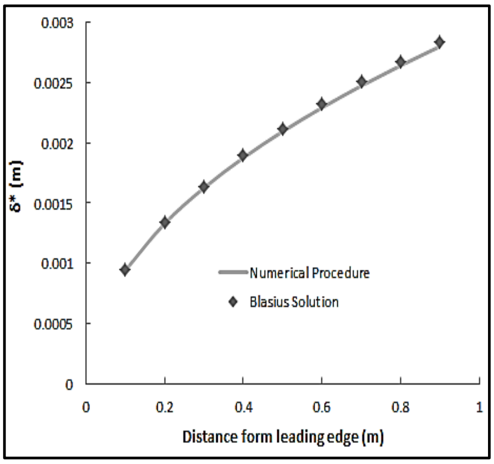


**Fig. 6.** Displacement thickness ($\delta^*$) at different position of the plate.

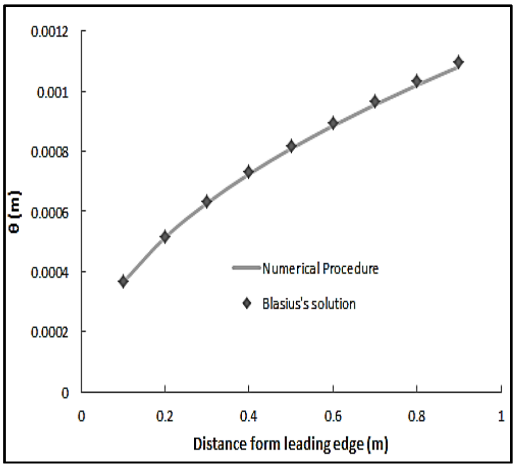


**Fig. 7.** Momentum thickness ($\theta$) at different position of the plate.

The shape factor from Blasius's calculation [2] is 2.59 for the flat plate while from present calculation, it is 2.590361 and as this value would be around 3.5 at separation [15], it can be concluded that separation of flow from the plate surface did not occur. The shear stress on a smooth plate is a direct function of the velocity gradient at the surface of the plate and this velocity gradient exists in a direction perpendicular to the surface. In immediate neighborhood of the body in which the velocity gradient normal to the wall is very large and the very small viscosity of the fluid exerts an essential influence that results in larger shear stress(Fig. 8). As we travel further upward from the plate, the influence of viscosity becomes trivial and flow at this region can be considered frictionless.

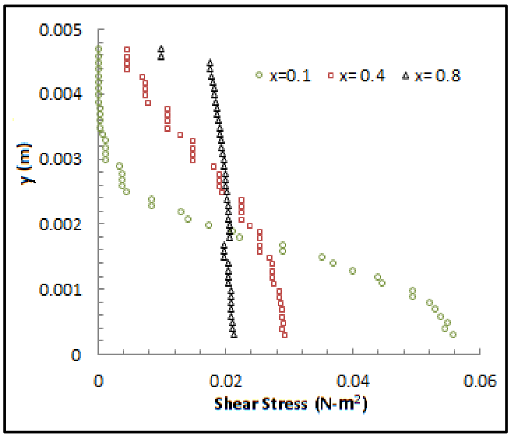


**Fig. 8.**Shear stress distribution in perpendicular direction of the plate.

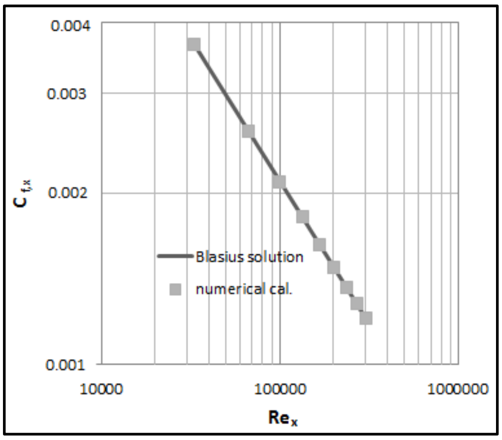


**Fig.9.** Friction coefficient ($C_{f,x}$) at different $Re_x$

As flow travels further downstream from the leading edge, Re increases and the velocity gradient decreases (in laminar boundary layer region) and thus the shear stress decreases. As the friction coefficient is directly proportional to the shear stress, it also decreases as the flow travels towards the downstream (Fig. 9). The skin friction drag for the smooth flat plate is found to be 0.002297 from Blasius's solution.

## 5. Conclusion

Different laminar boundary layer properties for air flow over a flat smooth plate is studied by Blasius's solution and Finite Volume Method of solving the Navier-Stokes equations and it is evident that the Finite Volume Method serves well in determining the behavior of laminar boundary layer. Different boundary layer thicknesses are direct function of local Re i.e. the distance from the leading edge of the plate and these thicknesses increases as the flow travels further downstream. Moreover the shear stress and subsequently, the local skin friction reduce as the flow travels further downstream. This analysis is limited for only one side of a smooth flat plate without any pressure gradient and heat generation or transfer phenomena and most importantly, when the flow is devoid of any turbulence.

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